

## ON A CONJECTURE OF LAUGESSEN AND MORPURGO

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ABSTRACT. A well known conjecture of R. Laugesen and C. Morpurgo asserts that the diagonal element of the Neumann heat kernel of the unit ball in  $\mathbb{R}^n$  ( $n \geq 1$ ) is a radially increasing function. In this paper, we use probabilistic arguments to settle this conjecture, and, as an application, we derive a new proof of the Hot Spots conjecture of J. Rauch in the case of the unit disk.

## 1. INTRODUCTION

We learned from Rodrigo Bañuelos the following conjecture of Richard Laugesen and Carlo Morpurgo which arose in connection with their work on conformal extremals of zeta functions of eigenvalues under Neumann boundary conditions in [LaMo]:

**Conjecture 1.1** (Laugesen-Morpurgo Conjecture). *Let  $p_U(t, x, y)$  denote the heat kernel for the Laplacian with Neumann boundary conditions on the unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$  in  $\mathbb{C}$ . For any  $t > 0$  arbitrarily fixed, the radial function  $p_U(t, x, x)$ , called the diagonal element of the heat kernel, is a strictly increasing function of  $|x|$ , that is,*

$$(1.1) \quad p_U(t, x, x) < p_U(t, y, y),$$

for any  $t > 0$  and  $x, y \in U$  with  $|x| < |y|$ .

Surprisingly, despite the seemingly simple nature of this conjecture and the fact that it seems to have been well known since 1994, we do not know of any progress on it, aside from some partial related results (see [PaNi], [PaPa1] and [PaPa2]). A more recent result related to this conjecture is due to Bañuelos et. al ([BaKuSi]), in which the authors show that the Laugesen-Morpurgo conjecture (1.1) holds for the transition density of the  $n$ -dimensional Bessel processes on  $(0, 1]$  reflected at 1 in the case  $n > 2$ , and that this is false for  $n = 2$ . Since the absolute value of a  $n$ -dimensional Brownian motion is a Bessel process of order  $n$ , this is equivalent to the monotonicity with respect to  $x \in (0, 1)$  of the integral mean

$$\int_0^{2\pi} p_U(t, x, xe^{i\theta}) d\theta.$$

*Remark 1.2.* The Laugesen–Morpurgo conjecture has the following physical interpretation. Consider a thermally insulated planar disk in which an atom of heat has been introduced at time  $t = 0$ . Then one feels “warmest” for all  $t > 0$  at the point where the atom of heat has been introduced if the point was chosen on the

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boundary of the disk, and feels “coldest” at this point for all  $t > 0$  if the chosen point was the center of the disk. Moreover, the corresponding temperature function (measured at the point where the atom of heat was inserted) is, for all  $t > 0$  arbitrarily fixed, a radially increasing function with respect to the point where the atom of heat was introduced. See the Theorem 4.4 and Remark 4.5 for a connection of the Laugesen-Morpurgo conjecture with the Hot Spots conjecture of Jeffrey Rauch.

In this paper, we use probabilistic arguments (couplings of reflecting Brownian motions) to settle the Laugesen-Morpurgo conjecture, first in the 2-dimensional case (Section 3), then in the general case (Section 4). The paper is organized as follows: in Section 2 we present the mirror coupling introduced by Burdzy et al. ([BuKe], and more recently [AtBu1], [AtBu2]), we establish the notation and we derive some properties of the coupling needed for the proof in the particular case of the reflecting Brownian motions in the unit disk.

In Section 3, in the particular case of the unit disk, we describe the motion of the mirror of the coupling (i.e. the line of the symmetry between the two processes), more precisely we show that for a certain choice of the starting points, the mirror always moves away from its starting point, towards the origin. This geometric property of the coupling, together with some probabilistic and analytic arguments allows us to settle the Laugesen-Morpurgo conjecture in the 2-dimensional case (Theorem 3.7).

In Section 4 we show that the arguments used in the previous section can be extended to obtain a proof of the general Laugesen-Morpurgo conjecture (we first present the 1-dimensional case, then we give the proof of the general case for  $n = 3$ , since it is easier to follow geometrically and notation-wise, yet it contains all the ideas involved in the proof of the general case), in Theorem 4.1.

As an application, we derive a different proof of the Hot Spots conjecture of Jeffrey Rauch, which suggests a possibly different approach for a resolution of this later conjecture, solved only partially at the present moment.

## 2. PRELIMINARIES

Our proof of Laugesen-Morpurgo conjecture relies on a certain property of the mirror coupling of reflecting Brownian motions in the unit disk and a representation of the Neumann heat kernel as an occupation time density of reflecting Brownian motion. We begin with a presentation of these results.

We denote by  $U = \{z \in \mathbb{C} : |z| < 1\}$  the unit disk in  $\mathbb{C}$  (which we identify with  $\mathbb{R}^2$ ).

We define the reflecting Brownian motion in  $U$  as a solution of the stochastic differential equation:

$$(2.1) \quad X_t = X_0 + B_t + \int_0^t \nu(X_s) dL_s, \quad t \geq 0.$$

Formally we have:

**Definition 2.1.**  $X_t$  is a reflecting Brownian motion in  $U$  starting at  $x_0 \in \overline{U}$  if it satisfies (2.1), where:

(a)  $B_t$  is a 2-dimensional Brownian motion started at 0 with respect to a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ ,

(b)  $L_t$  is a continuous nondecreasing process which increases only when  $X_t \in \partial U$ , i.e.  $\int_0^\infty 1_U(X_t) dL_t = 0$ , a.s.

(c)  $X_t$  is  $(\mathcal{F}_t)$ -adapted, and almost surely  $X_0 = x_0$  and  $X_t \in \overline{U}$  for all  $t \geq 0$ .

*Remark 2.2.* For pathwise existence and uniqueness of reflecting Brownian motion in the sense of the above definition see for example [BaHs].

Krzysztof Burdzy et. al ([BuKe], and more recently [AtBu1], [AtBu2]) introduced the mirror coupling of reflecting Brownian motions  $X_t$  and  $Y_t$  in a smooth planar domain  $D \subset \mathbb{R}^2$ , which we will describe briefly below.

The idea of the coupling is that the two processes behave like ordinary Brownian motion (symmetric with respect to a line of symmetry, called the mirror of the coupling) when both of them are inside the domain  $D$ . When one of the processes hits the boundary, the mirror  $M_t$  gets a (minimal) push towards the inward unit normal at the corresponding point at the boundary, needed in order to keep both processes in  $\bar{D}$ .

Considering the coupling time  $\tau = \inf \{t \geq 0 : X_t \in M_t\}$ , the mirror coupling evolves as described above for  $t < \tau$ , and we let  $X_t = Y_t$  for  $t \geq \tau$  (the two processes move together after the coupling time). For definiteness, for  $t \geq \tau$  we define the mirror  $M_t$  as the line passing through  $X_t = Y_t$  and making the same angle as  $M_\tau$  with the horizontal axis.

Formally, given two arbitrarily fixed points  $x, y \in U$ , we define the mirror coupling of reflecting Brownian motions in the unit disk, as a pair  $(X_t, Y_t)_{t \geq 0}$  of stochastic processes on a common filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ , given by

$$(2.2) \quad \begin{cases} X_t = x + W_t + \int_0^t \nu(X_s) dL_s^X \\ Y_t = y + Z_t + \int_0^t \nu(Y_s) dL_s^Y \end{cases}$$

where  $W_t$  is a 2-dimensional Brownian motion starting at  $W_0 = 0$ ,  $(\mathcal{F}_t)_{t \geq 0}$  is the filtration generated by the Brownian motion  $W_t$ ,  $Z_t$  is the mirror image of the Brownian motion  $W_t$  with respect to the line of symmetry  $M_t$  between  $X_t$  and  $Y_t$ , that is

$$(2.3) \quad Z_t = W_t - 2 \int_0^t \frac{X_s - Y_s}{\|X_s - Y_s\|^2} (X_s - Y_s) \cdot dW_s,$$

$L_t^X$  and  $L_t^Y$  denote the boundary local times of the reflecting Brownian motions  $X_t$  and respectively  $Y_t$ , and  $\nu(z) = -z$ ,  $z \in \partial U$ , denotes the inward unit normal vector field on the boundary of  $U$ . The processes  $X_t$  and  $Y_t$  evolve according to (2.2) above for  $t < \tau$ , where  $\tau$  is the coupling time (possibly infinite)

$$\tau = \inf \{t > 0 : X_t = Y_t\} \in \mathbb{R} \cup \{\infty\},$$

and the two processes evolve together (i.e.  $X_t = Y_t$ ) after they have coupled.

Setting

$$(2.4) \quad \begin{cases} X_t - Y_t = (m_t, n_t) \\ X_t + Y_t = (p_t, q_t) \end{cases}, \quad t \geq 0,$$

we have

$$\begin{cases} m_t = x^1 - y^1 + W_t^1 - Z_t^1 - \int_0^t X_s^1 dL_s^X + \int_0^t Y_s^1 dL_s^Y \\ n_t = x^2 - y^2 + W_t^2 - Z_t^2 - \int_0^t X_s^2 dL_s^X + \int_0^t Y_s^2 dL_s^Y \\ p_t = x^1 + y^1 + W_t^1 + Z_t^1 - \int_0^t X_s^1 dL_s^X - \int_0^t Y_s^1 dL_s^Y \\ q_t = x^2 + y^2 + W_t^2 + Z_t^2 - \int_0^t X_s^2 dL_s^X - \int_0^t Y_s^2 dL_s^Y \end{cases},$$

for all  $t < \tau$ , where the superscript 1 or 2 indicates the first, respectively the second component of the given point (when necessary, in order to avoid confusion with powers, we will use parantheses in order to indicate the square of a number).

Using the definition (2.3) of  $Z_t$ , we obtain

$$(2.5) \quad m_t = x^1 - y^1 + 2 \int_0^t \frac{m_s}{m_s^2 + n_s^2} (m_s dW_s^1 + n_s dW_s^2) - \int_0^t X_s^1 dL_s^X + \int_0^t Y_s^1 dL_s^Y,$$

and therefore the quadratic variation of  $M_t$  is given by

$$\langle m \rangle_t = 4 \int_0^t \frac{m_s^2}{m_s^2 + n_s^2} ds.$$

Similarly, we can obtain the following

$$(2.6) \quad \begin{cases} n_t = x^2 - y^2 + 2 \int_0^t \frac{n_s}{m_s^2 + n_s^2} (m_s dW_s^1 + n_s dW_s^2) - \int_0^t X_s^2 dL_s^X + \int_0^t Y_s^2 dL_s^Y \\ p_t = x^1 + y^1 + 2 \int_0^t \frac{n_s}{m_s^2 + n_s^2} (n_s dW_s^1 - m_s dW_s^2) - \int_0^t X_s^1 dL_s^X - \int_0^t Y_s^1 dL_s^Y \\ q_t = x^2 + y^2 + 2 \int_0^t \frac{n_s}{m_s^2 + n_s^2} (-n_s dW_s^1 + m_s dW_s^2) - \int_0^t X_s^1 dL_s^X - \int_0^t Y_s^1 dL_s^Y \end{cases},$$

and the corresponding quadratic variation processes:

$$(2.7) \quad \begin{cases} \langle m \rangle_t = \langle q \rangle_t = 4 \int_0^t \frac{m_s^2}{m_s^2 + n_s^2} ds \\ \langle n \rangle_t = \langle p \rangle_t = 4 \int_0^t \frac{n_s^2}{m_s^2 + n_s^2} ds \\ \langle m, n \rangle_t = -\langle p, q \rangle_t = 4 \int_0^t \frac{m_s n_s}{m_s^2 + n_s^2} ds \\ \langle m, p \rangle_t = \langle m, q \rangle_t = \langle n, p \rangle_t = \langle n, q \rangle_t = 0 \end{cases}$$

### 3. MAIN RESULTS

For  $t < \tau$ , the equation of the line of symmetry  $M_t$  between  $X_t$  and  $Y_t$  is given by

$$\left( z - \frac{X_t + Y_t}{2} \right) \cdot (X_t - Y_t) = 0,$$

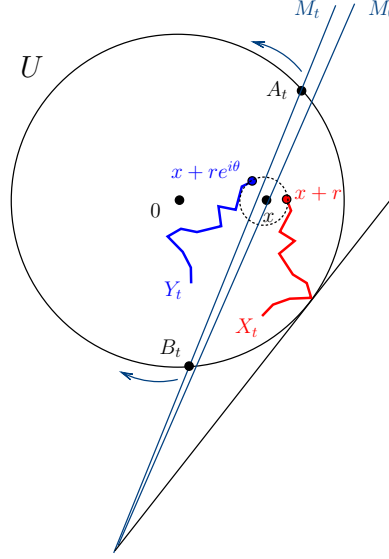
or equivalent

$$m_t u + n_t v - \frac{1}{2} (m_t p_t + n_t q_t) = 0,$$

where  $z = (u, v)$ .

The intersection of the mirror  $M_t$  with the boundary of  $U$  consists of two points  $A_t = (a_t^1, a_t^2)$  and  $B_t = (b_t^1, b_t^2)$ . The idea of the proof is that the mirror  $M_t$  moves to the left, in such a way that  $a_t^1$  and  $b_t^1$  are always decreasing (see Figure 1).

To show this, we consider the stopping time  $\tau_1 = \inf \{t > 0 : 0 \in M_t\}$ , and we consider the processes

FIGURE 1. The mirror coupling of reflecting Brownian motions in  $U$ .

$$(3.1) \quad \begin{cases} u_t = \frac{2m_t}{m_t p_t + n_t q_t} \\ v_t = \frac{2n_t}{m_t p_t + n_t q_t} \end{cases}, \quad t < \tau \wedge \tau_1.$$

Note that for  $t < \tau \wedge \tau_1$  we have

$$m_t p_t + n_t q_t = (X_t - Y_t) \cdot (X_t + Y_t) = |X_t|^2 - |Y_t|^2 \neq 0$$

and also

$$m_t^2 + n_t^2 = \|X_t - Y_t\|^2 \neq 0,$$

so the formulae above are well defined.

**Lemma 3.1.** *For  $t < \tau \wedge \tau_1$ ,  $u_t$  and  $v_t$  defined above are processes of bounded variation, explicitly given by*

$$du_t = \frac{2}{m_t p_t + n_t q_t} \left( u_t - \frac{m_t + p_t}{2} \right) dL_t^X$$

and

$$dv_t = \frac{2}{m_t p_t + n_t q_t} \left( v_t - \frac{n_t + q_t}{2} \right) dL_t^X$$

*Proof.* Applying the Itô formula to the  $C^2$  function  $f(m, n, p, q) = \frac{m}{mp+nq}$  and processes  $m_t, n_t, p_t$  and  $q_t$ , we have

$$\begin{aligned} \frac{1}{2}du_t &= d\left(\frac{m_t}{m_t p_t + n_t q_t}\right) \\ &= \frac{1}{(m_t p_t + n_t q_t)^2} (n_t q_t dm_t - m_t q_t dn_t - m_t^2 dp_t - m_t n_t dq_t) \\ &\quad + \frac{1}{2(m_t p_t + n_t q_t)^3} (-2n_t p_t q_t d\langle m \rangle_t + 2m_t q_t^2 d\langle n \rangle_t + 2m_t^3 d\langle p \rangle_t + 2m_t n_t^2 d\langle q \rangle_t) \\ &\quad + \frac{1}{2(m_t p_t + n_t q_t)^3} (2(m_t p_t q_t - n_t q_t^2) d\langle m, n \rangle_t + 4m_t^2 n_t d\langle p, q \rangle_t). \end{aligned}$$

Using the relations (2.5) and (2.6) it can be seen that the martingale part in the last expression above reduces to zero, and using (2.7) we obtain

$$\frac{1}{2}du_t = \frac{1}{(m_t p_t + n_t q_t)^2} \left[ -\frac{(m_t + p_t)}{2} (n_t q_t - m_t^2) + m_t \frac{(n_t + q_t)^2}{2} \right] dL_t^X.$$

Using the fact that  $L_t^X$  increases only when  $X_t \in \partial U$ , that is only when  $|X_t|^2 = \frac{(n_t + q_t)^2}{4} + \frac{(m_t + p_t)^2}{4} = 1$ , we obtain

$$\begin{aligned} \frac{1}{2}du_t &= \frac{1}{(m_t p_t + n_t q_t)^2} \left[ 2m_t - \frac{m_t + p_t}{2} (m_t p_t + n_t q_t) \right] dL_t^X \\ &= \frac{1}{m_t p_t + n_t q_t} \left( u_t - \frac{m_t + p_t}{2} \right) dL_t^X. \end{aligned}$$

A similar proof gives the expression for the process  $v_t$ , concluding the proof.  $\square$

Next, we prove that the mirror coupling in the unit disk leaves invariant the “left” and “right” positions of the starting points of the coupling, as follows:

**Lemma 3.2.** *If  $X_t, Y_t$  is a mirror coupling of reflecting Brownian motions in  $U$  with starting points  $X_0 = x + r$ ,  $Y_0 = x + re^{i\theta}$ ,  $x \in (0, 1)$ ,  $r \in (0, \min\{x, 1 - x\})$  and  $\theta \in [0, 2\pi)$ , then for all time  $t < \tau \wedge \tau_1$  the mirror  $M_t$  moves away from the point  $x$ , in such a way that  $a_t^1$  and  $b_t^1$  are non-increasing functions of  $t$ , where  $A_t = (a_t^1, a_t^2)$  and  $B_t = (b_t^1, b_t^2)$  are the points of intersection of  $M_t$  with the boundary of  $U$  (see Figure 1).*

*Proof.* Assuming the contrary, there exists a point  $P = (\alpha, \beta) \in U$  and times  $0 < t_1 < t_2 < \tau \wedge \tau_1$  such that  $P$  is to the right of  $M_{t_1}$  and to the left of  $M_{t_2}$ , that is we have

$$\begin{cases} m_{t_1}\alpha + n_{t_1}\beta - \frac{1}{2}(m_{t_1}p_{t_1} + n_{t_1}q_{t_1}) > 0 \\ m_{t_2}\alpha + n_{t_2}\beta - \frac{1}{2}(m_{t_2}p_{t_2} + n_{t_2}q_{t_2}) < 0 \end{cases}$$

or equivalent

$$\begin{cases} \alpha u_{t_1} + \beta v_{t_1} > 1 \\ \alpha u_{t_2} + \beta v_{t_2} < 1 \end{cases}$$

Setting  $t_0 = \inf \{t > t_1 : \alpha u_t + \beta v_t < 1\} \in (t_1, t_2)$ , from the previous lemma we obtain

$$\begin{aligned} & \alpha u_{t_0} + \beta v_{t_0} \\ = & \alpha u_{t_1} + \beta v_{t_1} + \int_{t_1}^{t_0} \frac{2}{m_t p_t + n_t q_t} \left( \alpha u_t + \beta v_t - \left( \frac{m_t + p_t}{2} \alpha + \frac{n_t + q_t}{2} \beta \right) \right) dL_t^X \\ \geq & \alpha u_{t_1} + \beta v_{t_1} \\ > & 1, \end{aligned}$$

since  $\alpha u_t + \beta v_t \geq 1$  for  $t \in [t_1, t_0]$  and

$$\left| \left( \frac{m_t + p_t}{2} \alpha + \frac{n_t + q_t}{2} \beta \right) \right| = |\alpha X_t^1 + \beta X_t^2| \leq |X_t| |(\alpha, \beta)| < 1.$$

By the continuity of the processes  $u_t$  and  $v_t$  we also must have  $\alpha u_{t_0} + \beta v_{t_0} = 1$ , a contradiction, which proves the claim.  $\square$

From the previous lemma we obtain the following:

**Theorem 3.3.** *For any  $x \in (0, 1)$ ,  $r, \varepsilon \in (0, \min \{x, 1 - x\})$  and  $\theta \in [0, 2\pi)$*

$$(3.2) \quad E^{x+re^{i\theta}} 1_{B(x, \varepsilon)}(Y_t) \leq E^{x+r} 1_{B(x, \varepsilon)}(X_t), \quad t \geq 0.$$

*Proof.* Let  $X_t, Y_t$  be a mirror coupling of reflecting Brownian motions in  $U$  with starting points  $X_0 = x + r$ ,  $Y_0 = x + re^{i\theta}$ ,  $x \in (0, 1)$ ,  $r, \varepsilon \in (0, \min \{x, 1 - x\})$  and  $\theta \in [0, 2\pi)$ . From the previous lemma it follows that the mirror  $M_t$  does not separate the points  $x$  and  $X_t$  (hence it separates the points  $x$  and  $Y_t$ ) for  $t < \tau \wedge \tau_1$ .

Since for  $t \geq \tau \wedge \tau_1$  either the two processes  $X_t$  and  $Y_t$  are symmetric with respect to the (fixed) line  $M_{\tau \wedge \tau_1}$  passing through the origin (for  $t \in (\tau \wedge \tau_1, \tau)$ ) or they have coupled (for  $t \in (\tau, \infty)$ ), it follows that in this case  $Y_t$  cannot reach  $B(x, \varepsilon)$  before coupling with  $X_t$ , and combining with the previous case we obtain that for all times  $t \geq 0$ ,  $Y_t \in B(x, \varepsilon)$  implies  $X_t \in B(x, \varepsilon)$ , and the claim follows.  $\square$

Denoting by  $p_U(t, x, y)$  the Neumann heat kernel for the unit disk (or equivalently, the transition density of reflecting Brownian motion in  $U$ ), we have the following:

**Lemma 3.4.** *For any  $t > 0$  and  $x, y \in U$  we have the following representation formula*

$$(3.3) \quad p_U(t, x, y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{|B(y, \varepsilon)|} E^x 1_{B(y, \varepsilon)}(W_t),$$

where  $W_t$  is a reflecting Brownian motion in the unit disk  $U$  with  $W_0 = x$  and  $E^x$  denotes the corresponding expectation with respect to the probability measure  $P^x$ .

*Proof.* Follows from the continuity of  $p_U(t, x, y)$ , see for example [PaPa1] for a proof.  $\square$

Combining Theorem 3.3 and Lemma 3.4, we obtain

**Proposition 3.5.** *For any  $x \in (0, 1)$ ,  $r \in (0, \min \{x, 1 - x\})$  and  $\theta \in [0, 2\pi)$  we have*

$$(3.4) \quad p_U(t, x + re^{i\theta}, x) \leq p_U(t, x + r, x), \quad t > 0.$$

*Remark 3.6.* The previous inequality can be interpreted as an extremal property of Brownian motion as follows:

$$\max_{y \in \partial B(x, r)} p_U(t, x, y) = p_U(t, x, x + r), \quad t > 0,$$

that is, among all reflecting Brownian motions in the unit disk with starting points on the circle  $\partial B(x, r)$ , the Brownian motion starting closest to the boundary (i.e. at the point  $x + r$ ) is most likely to return to (a neighborhood of)  $x$ . We will see that this property will allow us to prove the desired monotonicity in the Laugesen-Morpurgo conjecture.

We can now prove the main monotonicity property, as follows:

**Theorem 3.7.** *For any  $x \in (0, 1)$  and  $r \in (0, \min\{x, 1 - x\})$  we have*

$$(3.5) \quad \frac{1}{2\pi} \int_0^{2\pi} p_U(t, x + re^{i\theta}, x) d\theta \leq p_U(t, x + r, x) \leq p_U(t, x + r, x + r), \quad t > 0.$$

*Proof.* The first inequality follows by integrating the inequality (3.4) with respect to  $\theta \in [0, 2\pi)$ .

The second inequality follows by an argument similar to the one in Proposition 3.5 (see for example [PaPa1]).  $\square$

As a corollary of the above theorem, we obtain the following:

**Theorem 3.8** (Laugesen-Morpurgo conjecture). *The diagonal element of the Neumann heat kernel of the unit disk is a radially increasing function, that is*

$$p_U(t, x, x) < p_U(t, y, y),$$

for any  $t > 0$  and any  $x, y \in (0, 1)$  with  $x < y$ .

*Proof.* Let  $x \in (0, 1)$  and  $t > 0$  be arbitrarily fixed. From Theorem 3.7 we obtain

$$\begin{aligned} p_U(t, x + r, x + r) - p_U(t, x, x) &\geq \frac{1}{2\pi} \int_0^{2\pi} p_U(t, x + re^{i\theta}, x) d\theta - p_U(t, x, x) \\ &= \frac{1}{2\pi} \int_0^{2\pi} p_U(t, x + re^{i\theta}, x) - p_U(t, x, x) d\theta, \end{aligned}$$

for any  $r \in (0, \min\{x, 1 - x\})$ .

Dividing by  $r$  and passing to the limit with  $r \searrow 0$ , we obtain:

$$\begin{aligned} \frac{d}{dx} p_U(t, x, x) &= \lim_{r \searrow 0} \frac{p_U(t, x + r, x + r) - p_U(t, x, x)}{r} \\ &\geq \frac{1}{2\pi} \lim_{r \searrow 0} \int_0^{2\pi} \frac{p_U(t, x + re^{i\theta}, x) - p_U(t, x, x)}{r} d\theta. \end{aligned}$$

By bounded convergence theorem ( $p_U(t, \cdot, x)$  is a  $C^2$  function in the second variable, hence  $\nabla p_U(t, \cdot, x)$  is bounded in a neighborhood of  $x$ ), we obtain

$$\frac{d}{dx} p_U(t, x, x) \geq \frac{1}{2\pi} \int_0^{2\pi} \nabla p_U(t, x, x) \cdot e^{i\theta} d\theta = 0,$$

where we denoted by  $\nabla p_U$  the gradient of  $\nabla p_U(t, \cdot, x)$  in the second variable.

Since  $x \in (0, 1)$  was arbitrarily chosen, we have

$$\frac{d}{dx} p_U(t, x, x) \geq 0, \quad x \in (0, 1),$$



which shows that  $p_U(t, x, x)$  is an increasing function of  $x \in (0, 1)$  for any  $t > 0$  arbitrarily fixed.

Since  $p_U(t, x, x)$  is the diagonal of a heat kernel of an operator with real analytic coefficients implies that  $p_U(t, x, x)$  is a real analytic function, and therefore it cannot be constant on a non-empty subset of  $(0, 1)$ . This, together with the fact that  $p_U(t, x, x)$  is increasing on  $(0, 1)$  shows that  $p_U(t, x, x)$  is in fact strictly increasing for  $x \in (0, 1)$  for any  $t > 0$  arbitrarily fixed, concluding the proof.  $\square$

#### 4. EXTENSIONS AND APPLICATIONS

The Laugesen-Morpurgo Conjecture 1.1 has a natural extension to  $\mathbb{R}^n$  for any  $n \in \mathbb{N}^*$ , as follows:

**Conjecture 4.1** (General Laugesen-Morpurgo conjecture). *For  $n \in \mathbb{N}^*$  arbitrarily fixed, let  $p_{\mathbb{U}}(t, x, y)$  denote the heat kernel for the Laplacian with Neumann boundary conditions on the unit ball  $\mathbb{U} = \{x \in \mathbb{R}^n : \|x\| < 1\}$  in  $\mathbb{R}^n$ . For any  $t > 0$  arbitrarily fixed, the radial function  $p_{\mathbb{U}}(t, x, x)$ , called the diagonal element of the heat kernel, is a strictly increasing function of  $\|x\|$ , that is,*

$$(4.1) \quad p_{\mathbb{U}}(t, x, x) < p_{\mathbb{U}}(t, y, y),$$

for any  $t > 0$  and  $x, y \in \mathbb{U}$  with  $\|x\| < \|y\|$ .

We can use the ideas of the previous section (the case  $n = 2$ ) to prove the above conjecture for any  $n \in \mathbb{N}^*$ , as follows.

In the case  $n = 1$ , for  $0 < x < 1$  and  $0 < r < \min\{x, 1 - x\}$  arbitrarily fixed, the mirror coupling with starting points  $x - r$  and  $x + r$  given by (2.2)-(2.3) becomes

$$(4.2) \quad \begin{cases} X_t = x - r + W_t + \int_0^t \nu(X_s) dL_s^X \\ Y_t = x + r - W_t + \int_0^t \nu(Y_s) dL_s^Y \end{cases}$$

where  $W_t$  is a 1-dimensional Brownian motion starting at  $W_0 = 0$ .

Denoting by  $M_t$  the midpoint between  $X_t$  and  $Y_t$ , we have

$$M_t = \frac{X_t + Y_t}{2} = x + \frac{1}{2} \int_0^t \nu(X_s) dL_s^X + \frac{1}{2} \int_0^t \nu(Y_s) dL_s^Y,$$

and since  $L_s^X$  is constant for  $s < \tau_1 = \inf\{s > 0 : X_s = -1\}$ , and for  $s < \tau = \inf\{s > 0 : X_s = Y_s\}$  the process  $L_s^Y$  can increase only if  $\nu(Y_s) = \nu(1) = -1$  (the process  $Y_s$  cannot hit the boundary point  $-1$  unless it couples with  $X_s$  first), it follows that

$$M_{t \wedge \tau \wedge \tau_1} = x - \frac{1}{2} L_{t \wedge \tau \wedge \tau_1}^Y, \quad t \geq 0,$$

which shows that  $M_t$  is decreasing on the interval  $(0, \tau \wedge \tau_1)$ , and therefore the process  $Y_t$  is closer to  $x$  than  $X_t$  for all  $t \in (0, \tau \wedge \tau_1)$ .

Since for  $t \geq \tau \wedge \tau_1$  either the processes  $X_t$  and  $Y_t$  move together (for  $t \in [\tau, \infty)$ ), or they are symmetric with respect to the origin (for  $t \in [\tau \wedge \tau_1, \tau]$ ), it follows that for all  $t > 0$  the process  $Y_t$  is always closer to the point  $x$  than the process  $X_t$ , and therefore we obtain

$$p_I(t, x - r, x) \leq p_I(t, x + r, x),$$

for all  $t > 0$ ,  $x \in (0, 1)$  and  $r \in (0, \min\{x, 1 - x\})$ , where  $p_I(t, x, y)$  denotes the transition density of reflecting Brownian motion on the interval  $I = (-1, 1)$ .

A similar proof shows that  $p_I(t, x + r, x) \leq p_I(t, x + r, x + r)$ , and therefore we obtain

$$p_I(t, x - r, x) \leq p_I(t, x + r, x) \leq p_I(t, x + r, x + r),$$

or equivalent

$$\frac{p_I(t, x - r, x) + p_I(t, x + r, x)}{2} \leq p_I(t, x + r, x) \leq p_I(t, x + r, x + r),$$

for all  $t > 0$ ,  $x \in (0, 1)$  and  $r \in (0, \min\{x, 1 - x\})$ , which corresponds to the double inequality (3.5) in the case  $n = 1$ .

Following the proof of Theorem 3.8, for  $x \in (0, 1)$  arbitrarily fixed we obtain

$$\begin{aligned} \frac{\partial}{\partial x} p_I(t, x, x) &= \lim_{r \searrow 0} \frac{p_I(t, x + r, x + r) - p_I(t, x, x)}{r} \\ &\geq \frac{1}{2} \lim_{r \searrow 0} \frac{p_I(t, x - r, x) - p_I(t, x, x)}{r} \\ &\quad + \frac{p_I(t, x + r, x) - p_I(t, x, x)}{r} \\ &= \frac{1}{2} (-p'_I(t, x, x) + p'_I(t, x, x)) \\ &= 0, \end{aligned}$$

where we denoted by  $p'_I$  the derivative of the function  $p_I(t, \cdot, x)$  in the second variable, concluding the proof in the 1-dimensional case.

*Remark 4.2.* The fact that the Laugesen-Morpurgo conjecture is true in the case  $n = 1$  is known (see for example [BaKuSi], Remark 5.4 for an analytic proof, or [PaPa2] for a different probabilistic proof). We presented it here for a unitary treatment of the Laugesen-Morpurgo conjecture, in order to show that the same argument can be applied regardless of the dimension  $n \in \mathbb{N}^*$ .

The coupling arguments in the previous section can also be applied, with the appropriate changes, to give a proof of the general Laugesen-Morpurgo Conjecture 4.1 in the case  $n \geq 3$ . For example, in the case  $n = 3$ , replacing in (2.2)-(2.3)  $W_t$  by a 3-dimensional Brownian motion starting at  $W_0 = 0$  and  $\nu$  by the the corresponding unit normal vector field on the boundary of the unit ball  $\mathbb{U} = \{x \in \mathbb{R}^3 : \|x\| < 1\}$ ,  $\nu_{\mathbb{U}}(x) = -x$ , where  $x = (x^1, x^2, x^3) \in \partial\mathbb{U}$ , the same formulae give the mirror coupling in  $\mathbb{U}$  with starting points  $X_0 = x \in \mathbb{U}$  and  $Y_0 = y \in \mathbb{U}$ .

Following Lemma 3.2, for distinct starting points  $X_0 = (x^1 + \rho, 0, 0)$ ,  $Y_0 = (x^1, 0, 0) + \rho u \in \mathbb{U}$  with  $x^1 \in (0, 1)$ ,  $\rho \in (0, \min\{1 - x^1, x^1\})$  and  $\|u\| = 1$ , we need to show that for  $t \leq \tau \wedge \tau_1$  ( $\tau$  and  $\tau_1$  are the coupling time, respectively the hitting time of the origin by  $M_t$ ), the plane of symmetry  $M_t$  between between  $X_t$  and  $Y_t$  given by

$$(X_t^1 - Y_t^1)z^1 + (X_t^2 - Y_t^2)z^2 + (X_t^3 - Y_t^3)z^3 - \frac{\|X_t\|^2 - \|Y_t\|^2}{2} = 0, \quad (z^1, z^2, z^3) \in \mathbb{R}^3,$$

moves away from the point  $M_0 = x$ , towards the origin (i.e. to the “left”).

Assuming the contrary, there exists a point  $P = (\alpha, \beta, \gamma) \in \mathbb{U}$  and times  $0 < t_1 < t_2 < \tau \wedge \tau_1$  such that  $P$  is to the right of  $M_{t_1}$  and to the left of  $M_{t_2}$ , that is

we have:

$$\begin{cases} (X_{t_1}^1 - Y_{t_1}^1) \alpha + (X_{t_1}^2 - Y_{t_1}^2) \beta + (X_{t_1}^3 - Y_{t_1}^3) \gamma - \frac{1}{2} (\|X_{t_1}\|^2 - \|Y_{t_1}\|^2) > 0 \\ (X_{t_2}^1 - Y_{t_2}^1) \alpha + (X_{t_2}^2 - Y_{t_2}^2) \beta + (X_{t_2}^3 - Y_{t_2}^3) \gamma - \frac{1}{2} (\|X_{t_2}\|^2 - \|Y_{t_2}\|^2) < 0 \end{cases}$$

or equivalent

$$\begin{cases} \alpha u_{t_1} + \beta v_{t_1} + \gamma w_{t_1} > 1 \\ \alpha u_{t_2} + \beta v_{t_2} + \gamma w_{t_2} < 1 \end{cases},$$

where

$$u_t = \frac{2(X_t^1 - Y_t^1)}{\|X_t\|^2 - \|Y_t\|^2}, \quad v_t = \frac{2(X_t^2 - Y_t^2)}{\|X_t\|^2 - \|Y_t\|^2}, \quad w_t = \frac{2(X_t^3 - Y_t^3)}{\|X_t\|^2 - \|Y_t\|^2}.$$

As in Lemma 3.1,  $u_t$ ,  $v_t$  and  $w_t$  are processes of bounded variation on  $(0, \tau \wedge \tau_1)$ , explicitly given by

$$du_t = \frac{2}{\|X_{t_1}\|^2 - \|Y_{t_1}\|^2} (u_t - X_t^1) dL_t^X,$$

where  $L_t^X$  is the local time of  $X_t$  on the boundary of  $\mathbb{U}$ , and similarly for  $v_t$  and  $w_t$ .

Setting  $t_0 = \inf \{t > t_1 : \alpha u_t + \beta v_t + \gamma w_t < 1\} \in (t_1, t_2)$ , we obtain

$$\begin{aligned} \alpha u_{t_0} + \beta v_{t_0} + \gamma w_{t_0} &= \alpha u_{t_1} + \beta v_{t_1} + \gamma w_{t_1} \\ &\quad + 2 \int_{t_1}^{t_0} \frac{\alpha u_t + \beta v_t + \gamma w_t - (\alpha X_t^1 + \beta X_t^2 + \gamma X_t^3)}{\|X_{t_1}\|^2 - \|Y_{t_1}\|^2} dL_t^X \\ &\geq \alpha u_{t_1} + \beta v_{t_1} + \gamma w_{t_1} \\ &> 1, \end{aligned}$$

since  $\alpha u_t + \beta v_t + \gamma w_t \geq 1$  for  $t \in [t_1, t_0]$  and

$$|\alpha X_t^1 + \beta X_t^2 + \gamma X_t^3| \leq \|X_t\| \|(\alpha, \beta, \gamma)\| \leq \|(\alpha, \beta, \gamma)\| < 1$$

for all  $t \geq 0$ .

By the definition of  $t_0$  and the continuity of the processes  $u_t$ ,  $v_t$  and  $w_t$  we must also have  $\alpha u_{t_0} + \beta v_{t_0} + \gamma w_{t_0} = 1$ , contradicting  $\alpha u_{t_0} + \beta v_{t_0} + \gamma w_{t_0} > 1$ , which proves the claim.

Using the previous argument and proceeding as in Theorem 3.3, we obtain

$$(4.3) \quad E^{(x^1, 0, 0) + \rho u} 1_{B((x^1, 0, 0), \varepsilon)}(Y_t) \leq E^{(x^1 + \rho, 0, 0)} 1_{B((x^1, 0, 0), \varepsilon)}(X_t),$$

for any  $t > 0$ ,  $x^1 \in (0, 1)$ ,  $\rho, \varepsilon \in (0, \min\{x^1, 1 - x^1\})$  and  $u \in \partial\mathbb{U}$ .

Using the equivalent result in Lemma 3.4 for the 3-dimensional reflecting Brownian motion, we obtain

$$p_{\mathbb{U}}(t, (x^1, 0, 0) + \rho u, (x^1, 0, 0)) \leq p_{\mathbb{U}}(t, (x^1 + \rho, 0, 0), (x^1, 0, 0)),$$

which as in Theorem 3.7 shows that for any  $t > 0$ ,  $x = (x^1, 0, 0) \in \mathbb{U}$ ,  $x^1 > 0$  and  $\rho \in (0, \min\{x^1, 1 - x^1\})$  we have

$$\begin{aligned} &\frac{1}{4\pi} \int_{\partial\mathbb{U}} p_{\mathbb{U}}(t, (x^1, 0, 0) + \rho u, (x^1, 0, 0)) d\sigma(u) \\ &\leq p_{\mathbb{U}}(t, (x^1 + \rho, 0, 0), (x^1, 0, 0)) \\ &\leq p_{\mathbb{U}}(t, (x^1 + \rho, 0, 0), (x^1 + \rho, 0, 0)), \end{aligned}$$

where  $d\sigma(u)$  is the surface measure on  $\partial\mathbb{U}$ .

Proceeding as in the proof of Theorem 3.8 and using the rotational invariance of reflecting Brownian motion in  $\mathbb{U}$ , we obtain

$$p_{\mathbb{U}}(t, x, x) < p_{\mathbb{U}}(t, y, y),$$

for any  $t > 0$  and any  $x, y \in \mathbb{U}$  with  $\|x\| < \|y\|$ , concluding the proof of the conjecture in the case  $n = 3$ .

The above discussion can be summarized in the following resolution of the general Laugesen-Morpurgo conjecture:

**Theorem 4.3.** *The diagonal element  $p_{\mathbb{U}}(t, x, x)$  of the Neumann heat kernel of the unit ball  $\mathbb{U}$  in  $\mathbb{R}^n$  ( $n \in \mathbb{N}^*$ ) is a radially increasing function, that is we have*

$$p_{\mathbb{U}}(t, x, x) < p_{\mathbb{U}}(t, y, y),$$

for any  $t > 0$  and any  $x, y \in \mathbb{U}$  with  $\|x\| < \|y\|$ .

The problem of monotonicity of Neumann heat kernel is closely related to the celebrated “Hot Spots” conjecture of Jeffrey Rauch, which asserts that the extrema of second Neumann eigenfunctions of a planar convex domain are attained only on the boundary of the domain. The Hot Spots Conjecture has been verified for several types of convex domains in the plane (see for example [Pa] and the references cited therein).

Using and eigenfunction expansion of the transition density of reflecting Brownian motion, it can be seen that the monotonicity in the Laugesen-Morpurgo conjecture implies a similar monotonicity of the second Neumann eigenfunction(s). As an application of Theorem 3.8, we can derive the Hot Spots conjecture in the case of the unit disk (see for example [Pa] for an alternate proof), as follows:

**Theorem 4.4.** *If  $\varphi$  is a second Neumann eigenfunction of the Laplaceian in the unit disk, then  $\varphi(x)$  is a radially monotonic function. In particular,  $\varphi$  attains its maximum and minimum only on the boundary of  $U$ , that is*

$$\min_{\partial U} \varphi < \varphi(x) < \max_{\partial U} \varphi, \quad x \in U,$$

which shows that the Hot Spots conjecture holds for the unit disk  $U$ .

*Proof.* Using an eigenfunction expansion of  $p_U(t, x, y)$ , it can be seen that for large  $t$  we have

$$p_U(t, x, x) \approx \frac{1}{\sqrt{\pi}} + e^{-\lambda_2 t} \varphi_2^2(x) + e^{-\lambda_2 t} \psi_2^2(y),$$

where  $\varphi_2(re^{i\theta}) = J_2(\sqrt{\lambda_2}r) \cos \theta$  and  $\psi_2(re^{i\theta}) = J_2(\sqrt{\lambda_2}r) \sin \theta$  are two independent second Neumann eigenfunctions for the Laplaceian on  $U$ ,  $\lambda_2$  is the second Neumann eigenvalue and  $J_2$  is the Bessel function of order 2 (see for example [Ba], pp. 92 – 93).

From Theorem 3.8 it follows that

$$\varphi_2^2(r) + \psi_2^2(r) = J_2^2(\sqrt{\lambda_2}r)$$

is an increasing function of  $r$ , and therefore for an arbitrary second Neumann eigenfunction  $\varphi$  (the second Neumann eigenspace is 2-dimensional) we have

$$\begin{aligned} \varphi(re^{i\theta}) &= \alpha \varphi_2(re^{i\theta}) + \beta \psi_2(re^{i\theta}) \\ &= J_2(\sqrt{\lambda_2}r) (\alpha \cos \theta + \beta \sin \theta) \end{aligned}$$

is a monotonic function of  $r$  for  $\theta \in [0, 2\pi)$  arbitrarily fixed, and the claim follows.  $\square$

*Remark 4.5.* The fact that the Hot Spots conjecture holds in the case of the unit disk is a known result (see for example [Ka], [Pa]). The above proof is meant to show the connection of the Laugesen-Morpurgo conjecture with the Hot Spots conjecture, connection which may be used for a possibly different approach in the resolution of the later conjecture.

More precisely, if the Laugesen-Morpurgo conjecture can be extended to a certain smooth convex domain  $D$ , that is, if it can be shown that the diagonal element  $p_D(t, x, x)$  of the transition density of the reflecting Brownian motion in  $D$  is an increasing function of  $x$  along a certain family of curves covering  $D$ , then, at least in the case of the 1-dimensional second Neumann eigenspace (it is known that the second Neumann eigenspace is either 1 or 2-dimensional), as in the above proof it follows that the second Neumann eigenfunctions are monotone along the same family of curves, thus proving that the Hot Spots conjecture holds for the domain  $D$ .

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